Path-Complete Lyapunov Techniques: stability, safety, and beyond*

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Path-complete Lyapunov Techniques[1] are a family of methods that combine Automata-Theoretic tools with algebraic formulas in order to derive ad hoc criteria for the control of complex systems. These criteria are typically solved with Convex Optimization solvers. They initially appeared in the framework of switched systems, which are dynamical systems for which the state dynamics varies between different operating modes. They take the form

$$x(t+1) = f_{\sigma(t)}(x(t)) \tag{1}$$

where the state $x(t)$ evolves in $\mathbb{R}^n$. The *mode* $\sigma(t)$ of the system at time t takes its value in a set $\{1,\dots,M\}$ for some integer M, and each mode of the system is described by a continuous map $f_i(x) : \mathbb{R}^n \to \mathbb{R}^n$.

When the functions f_i are linear functions, we say that the system is a *linear switched system.* The *stability problem* is reputedly very hard, even in the restricted case of linear functions (see e.g. [14, Section 2.2]). In this case, one can easily obtain a sufficient condition for stability, through the existence of a *common quadratic* Lyapunov function (see e.g. [18, Section II-A]). However, such a Lyapunov function may not exist, even when the system is asymptotically stable (see e.g. [17, 18]). Less conservative parameterizations of candidate Lyapunov functions have been proposed, at the cost of greater computational effort (e.g. for linear switching systems, [19] uses sum-of-squares polynomials, [12] uses max-of-quadratics Lyapunov functions, and [4] uses polytopic Lyapunov functions). *Multiple Lyapunov functions* (see [7, 21, 13]) arise as an alternative to common Lyapunov functions. In the case of linear systems, the multiple *quadratic* Lyapunov functions such as those introduced in [6, 8, 16, 9] hold special interest as checking for their existence boils down to solving a set of LMIs. The general framework of *Path-Complete* Lyapunov functions was recently introduced in [1, 15] in this context, for analyzing and unifying these approaches.

In this talk, we first present these criteria guaranteeing that the system (1) is stable under *arbitrary switching*, i.e. where the function $\sigma(\cdot)$ is not constrained, and one is interested in the worst-case stability. We then show how this very natural idea can be leveraged for much more general purposes: we present recent works were the same

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[1] Path-complete techniques are implemented in the JSR toolbox [22].

idea has been applied to more general systems than the ones described above [20], or for proving different properties than stability [10].

These techniques give rise to many natural questions: First, they essentially provide algebraic criteria, that is, equations and inequations, that can be solved numerically in order to (hopefully) conclude stability, if a solution is found. But what do they mean in terms of control systems? Do they have a geometric interpretation in the state space? Second, among the different criteria in this framework, which one should an engineer pick in practice? Do these criteria compare with each other (in terms of conservativeness)? How to algorithmically choose the good criterion, when one is given a particular problem? While recent progress has been done to provide a geometric interpretation of these criteria [3], several problems remain open, like the one of comparing two given path-complete criteria [2].

Finally, we draw connections with other recent works in Control and Computer Science, which bear similarities with path-complete techniques, in safety analysis of computer programs [5], or in connection with tropical Kraus maps [11].

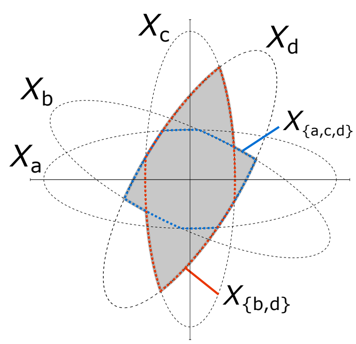

Fig. 1. Graphical illustration of the level set of a path-complete Lyapunov function. We will show in the talk that these level sets can always be expressed as unions of intersections of Ellipsoids.

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