Space-Efficient Fragments of Higher-Order Fixpoint Logic

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Motivation

Higher-order Modal Fixpoint Logic:

- extension of the modal μ-calculus by simply typed λ-calculus
- very expressive logic, e.g., context-sensitive path languages, assume-guarantee properties
- captures time hierarchy: type order $k+1$ model checking is $k+1$-EXPTIME complete (via alternating reachability game)

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- matching lower bound via reduction to order-k corridor tiling problem
- still very expressive: $a^n b^n c^n$ path language

Syntax of HFL

HFL $=$ modal μ-calculus

$$\varphi \quad ::= \quad P \mid X \qquad \mid \varphi \vee \varphi \mid \neg\varphi \mid \langle a \rangle \varphi \mid \mu\, X \qquad .\varphi$$

plus duals $\varphi \wedge \psi$, $[a]\varphi$, $\nu\, X$ *natively*

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HFL = modal μ-calculus + simply typed λ-calculus

[Viswanathan^2 '04]

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well-formedness condition given by type system

Types

simple types given via $\tau ::= \mathsf{Pr}$

complete lattice over transition system

$\mathcal{T} = (\mathcal{S}, \to, L)$

$[\![\mathsf{Pr}]\!] := (2^{\mathcal{S}}, \subseteq)$

Types

simple types given via $\tau ::= \mathsf{Pr} \mid \tau \to \tau$

because of right-associativity: $\tau = \tau_1 \to \ldots \to \tau_m \to \mathsf{Pr}$

each type induces a complete lattice over transition system $\mathcal{T} = (\mathcal{S}, \to, L)$ using pointwise orderings $\sqsubseteq$

$$\begin{aligned} [\![\mathsf{Pr}]\!] &:= (2^{\mathcal{S}}, \subseteq) \\ [\![\sigma \to \tau]\!] &:= ([\![\sigma]\!] \to [\![\tau]\!], \sqsubseteq) \end{aligned}$$

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Consider $(\mu X.\lambda x.\, x \vee (X\,[a]x))\,P$.

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We get: $P \vee [a]P \vee [a][a]P \vee \ldots = \bigvee_{i=0}^{n} [a]^i P$

uniform inevitability!

Tail-Recursion

Limit use of fp vars: No free fixpoint variables in all subformulas of certain form

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Thm.: The order $(k+1)$ model-checking problem for tail-recursive HFL is k-EXPSPACE-complete.

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- cover $\wedge, [a], \neg$, application via tail-recursion

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Model-checking whether state t in some LTS is model of ψ

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- ...
- If $\varphi = \varphi_1 \wedge \varphi_2$, $\rightarrow \varphi_1$ has no free fp vars. Return false if **check**$(s, \varphi_1, (f_1, \ldots, f_n), \eta, \mathrm{cnt})$ returns false, return **check**$(s, \varphi_2, (f_1, \ldots, f_n), \eta, \mathrm{cnt})$ else

The Upper Bound (cont.)

Behavior of **check**$(s, \varphi, (f_1, \ldots, f_n), \eta, \mathsf{cnt})$ depends on form of φ:

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- If $\varphi = \varphi'\,\varphi''$, $\to \varphi''$ has no free fp vars. Compute $f_{n+1} =$ **buildFT**(φ'', η), return **check**$(s, \varphi', (f_1, \ldots, f_n, f_{n+1}), \eta, \mathsf{cnt})$

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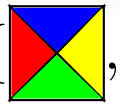
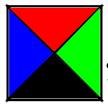
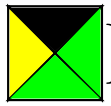
- $\ldots$
- If $\varphi = \mu(X : \tau).\,\varphi'$, return **check**$(s, \varphi', (f_1, \ldots, f_n), \eta, \mathsf{cnt}[X \mapsto \mathsf{ht}(\tau)])$.
- If $\varphi = X$, X least fp var, return false if $\mathsf{cnt}(X) = 0$, else return **check**$(s, \mathsf{fp}(X), (f_1, \ldots, f_n), \eta, \mathsf{cnt}[X\text{- - }]$.

buildFT(φ, η) calls **check**$(\ldots)$ again, iterating over all suitable arguments

The Order-k Corridor Tiling Problem

Instance $\mathcal{K}$ is set $T = \{$  ,  , $\ldots,$  $\}$ of tiles

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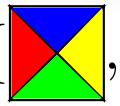
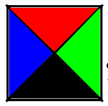
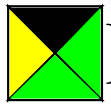
Instance $\mathcal{K}$ is set $T = \{$  ,  , $\ldots,$  $\}$ of tiles, horizontal and vertical matching relations H and V (indicated by colors)

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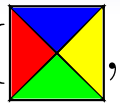
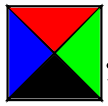
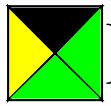
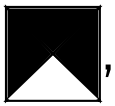
Instance $\mathcal{K}$ is set $T = \{$  ,  , $\ldots$,  $\}$ of tiles, horizontal and vertical matching relations H and V (indicated by colors), and initial tiles $t_I =$  and $T_\square =$  , and final tile $t_F =$ 

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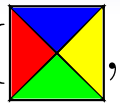
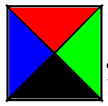
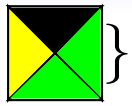
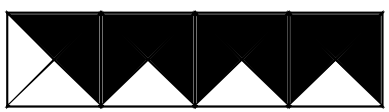
Instance $\mathcal{K}$ is set $T = \{$  ,  , $\ldots,$  $\}$ of tiles, horizontal and vertical matching relations H and V (indicated by colors), and initial tiles $t_I =$  and $T_\square =$  , and final tile $t_F =$ 

Start with initial 2^n_k wide row,



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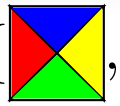
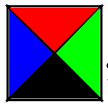
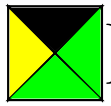
Instance $\mathcal{K}$ is set $T = \{$  ,  , $\ldots$,  $\}$ of tiles, horizontal and vertical matching relations H and V (indicated by colors), and initial tiles $t_I =$  and $T_\square =$  , and final tile $t_F =$ 

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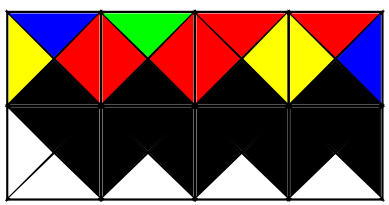

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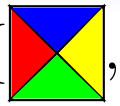
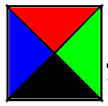
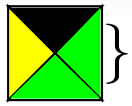
Instance $\mathcal{K}$ is set $T = \{$  $,$  $, \ldots,$  $\}$ of tiles, horizontal and vertical matching relations H and V (indicated by colors), and initial tiles $t_I =$  and $T_\square =$  , and final tile $t_F =$ 

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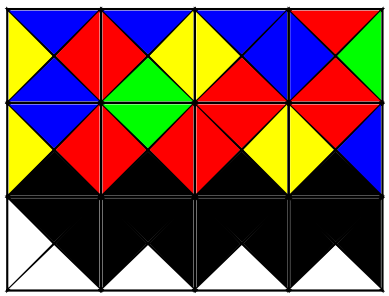

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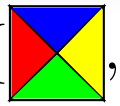
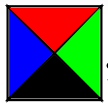
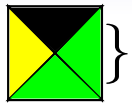
Instance $\mathcal{K}$ is set $T = \{$  ,  , $\ldots,$  $\}$ of tiles, horizontal and vertical matching relations H and V (indicated by colors), and initial tiles $t_I =$  and $T_\square =$  , and final tile $t_F =$ 

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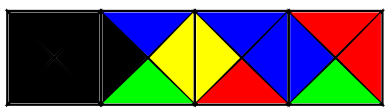

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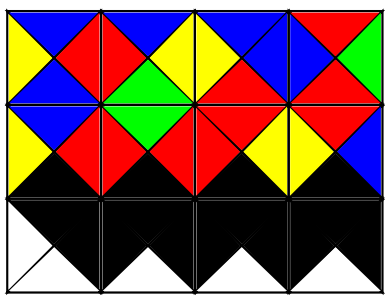

For $k \geq 0$, the order-k corridor tiling problem is k-EXPSPACE hard (van Emde Boas '97)

Jones' Encoding of Large Numbers

Observation: Numbers up to 2^n_{k+1} have at most 2^n_k bits in binary

Fix an LTS with states $\mathcal{S} = \{0, \ldots, n-1\}$

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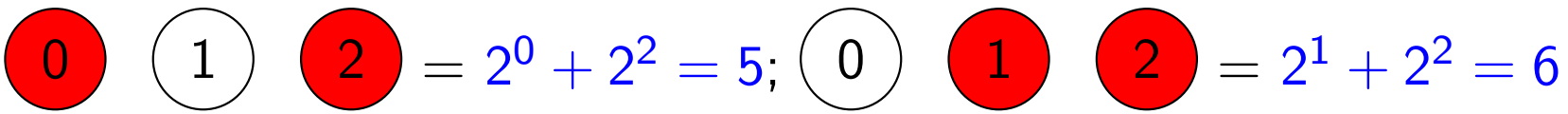

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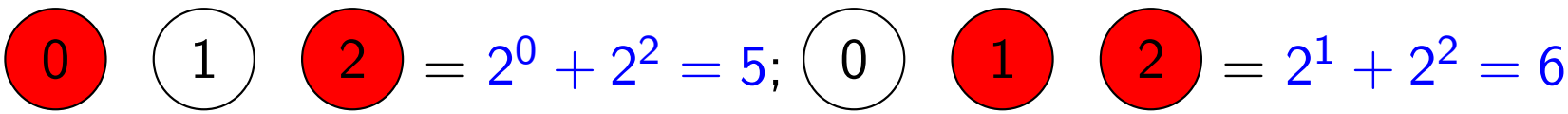
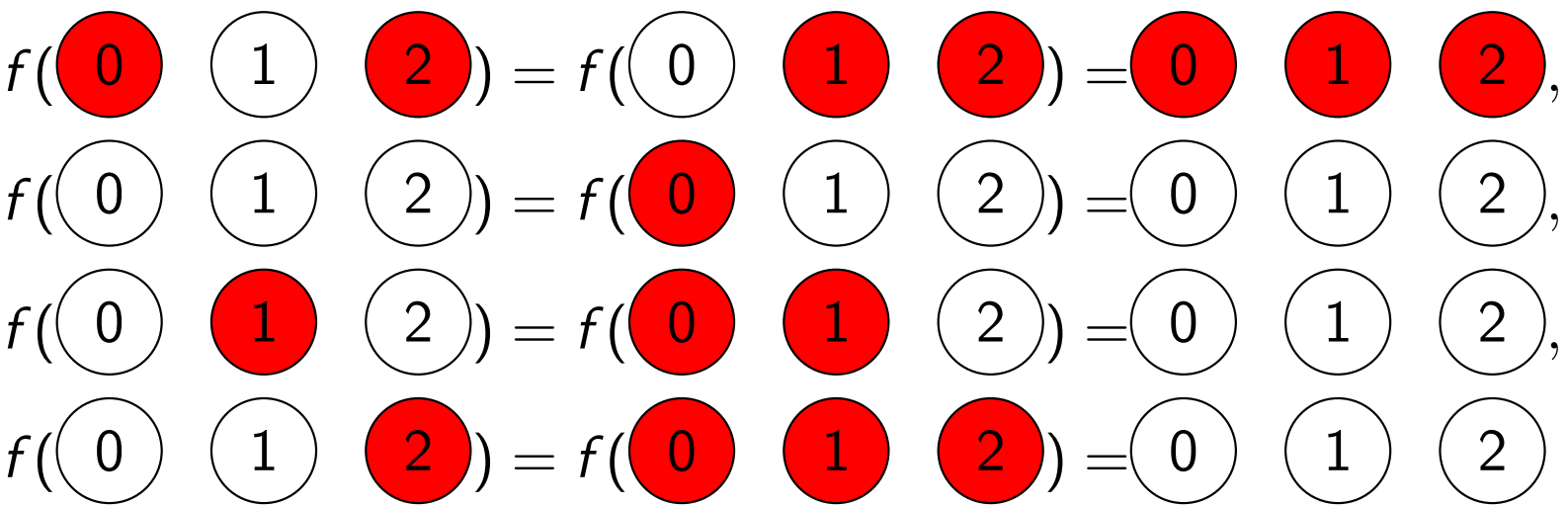


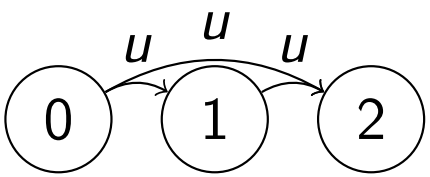
f with



$= 2^5 + 2^6 = 32 + 64 = 96$

Working with Jones Encodings

Add transitions u



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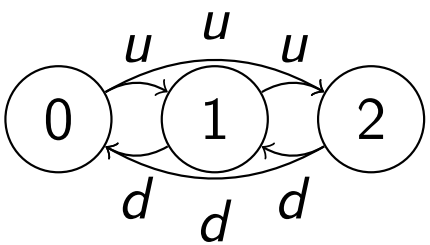
Add transitions u, d and e $= \mathcal{S} \times \mathcal{S}$.



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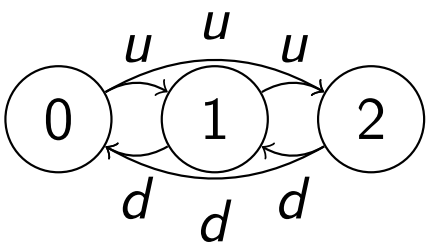

$\rightarrow$ Compare numbers: m_1 bigger than m_2 if ex. a bit set in m_1 not set in m_2, and all higher bits are set in m_1 if they are set in m_2:

$$\mathsf{gt}_0 = \lambda(m_1, m_2 : \tau_0).\ \langle \mathsf{e} \rangle \big(m_2 \wedge \neg m_1 \wedge [\mathsf{u}](m_1 \Rightarrow m_2) \big)$$

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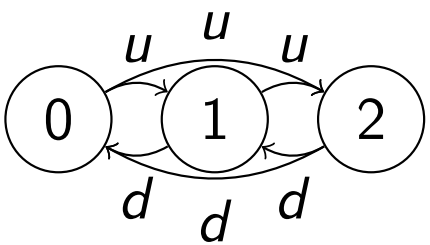

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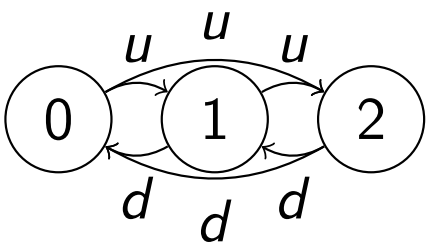

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$$\mathsf{find}_k = \lambda(p : \tau_{k+1}).$$
$$\Big(\big(\mu(F : \tau_k \to \mathsf{Pr})\,.\ \lambda(m : \tau_k)\,.\ ([\mathsf{e}](p\ m)) \vee F\ (\mathsf{inc}_k\ m) \big)\ \mathsf{zero}_k \Big)$$

Encoding the Corridor Tiling Problem

Fix corridor tiling problem $\mathcal{K} = (T, H, V, t_I, t_\square, t_F)$ and extend LTS from previous slides with propositions $p_I = \{0\}, p_\square = \{|T| - 2\}, p_F = \{|T| - 1\}$ and transitions $\mathsf{h} = \{(i,j) \colon (t_i, t_j) \in H\}$, $\mathsf{v} = \{(i,j) \colon (t_i, t_j) \in V\}$.

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Check for final row:

$\mathsf{isFinal} = \lambda(r : \tau_k).\ [\mathsf{e}]\big((r\ \mathsf{zero}_{k-1}) \Rightarrow p_F\big)$

The Lower Bound

Formula that encodes problem:
$\big(\mu(P\colon \tau_{k+1}).\, \lambda(r_1\colon \tau_k).(\mathsf{isFinal}\ r_1) \vee (\mathsf{exists\_succ}\ r_1\ P)\big)\ \mathsf{init}$

where $\mathsf{exists\_succ} =$
$\lambda(r_1\colon \tau_k, p\colon \tau_{k+1}).\ \mathsf{find}\ \big(\lambda(r_2\colon \tau_k).\ (\mathsf{horiz}\ r_2) \wedge (\mathsf{vert}\ r_1\ r_2) \wedge (p\ r_2).$
and horiz_k verifies horizontal matching in row, and
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Not tail-recursive, but can β-reduce:

$$\Big(\mu(P\colon \tau_{k+1}).\, \lambda(r_1\colon \tau_k).\quad (\dots) \vee \big(\mu(F\colon \tau_{k+1}).\lambda(r_2\colon \tau_k)\ ((\dots) \wedge (P\ r_2)) \vee (F\ (\text{inc } r_2))\big)\ r_1\Big)\ \text{init}$$

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