Distributed control synthesis using Euler's method

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Switched systems

A continuous switched system

$$\dot{x}(t) = f_{\sigma(t)}(x(t))$$

- state $x(t) \in \mathbb{R}^n$
- switching rule $\sigma(\cdot) : \mathbb{R}^+ \longrightarrow U$
- finite set of (switched) modes $U = \{1, \ldots, N\}$

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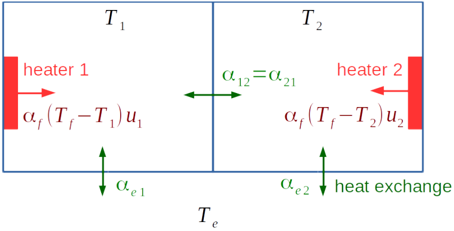
Control Synthesis problem:
Find at each sampling time, the appropriate mode $u \in U$ (in function of the value of $x(t)$) in order to make the system satisfy a certain property.

Example: Two-room apartment



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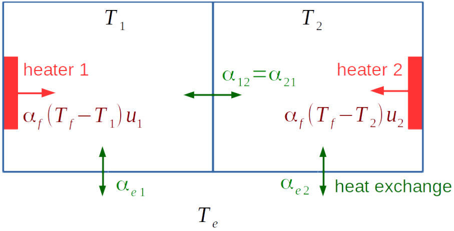

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- Modes: $\begin{pmatrix}u_1\\ u_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$; sampling period τ

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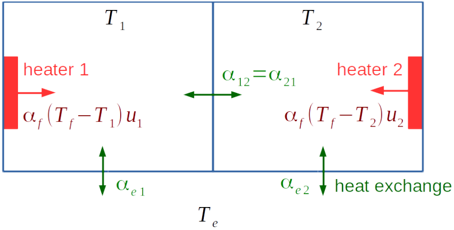

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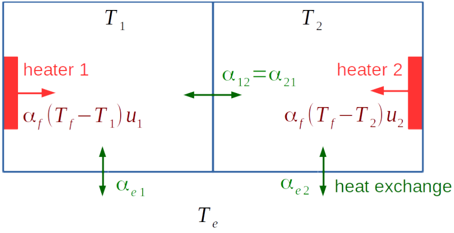

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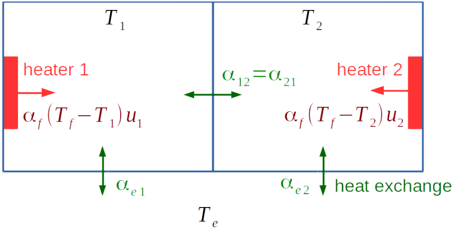

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- A state dependent control consists in selecting at each τ a mode (or a pattern) according to the current value of the state.

Reachability and Stability Problems

We consider the state-dependent control problem of synthesizing σ:

At each sampling time t, find the appropriate switched mode $u \in U$ according to the current value of x, in order to achieve some objectives:

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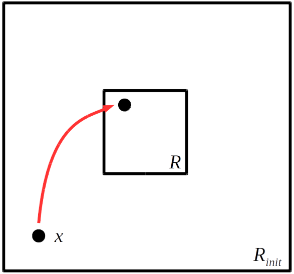

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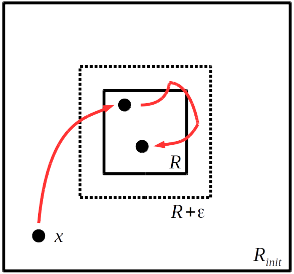

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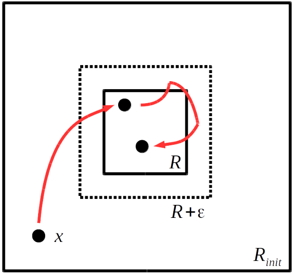

NB: classic stabilization to an equilibrium point, impossible to achieve here

$\rightsquigarrow$ practical stability

Focus on (R, S)-stability

Being given a recurrence (rectang.) set R and a safety (rectang.) set S, we consider the state-dependent control problem of synthesizing σ:

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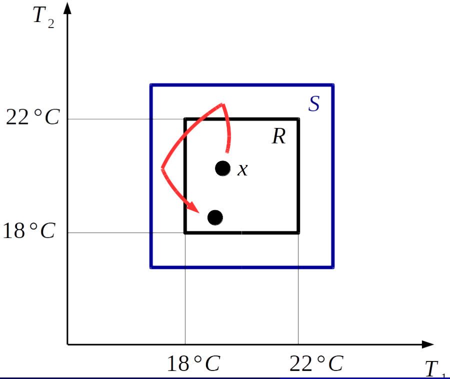

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if $x(0) \in R$, then $x(t)$:

1. returns infinitely often into R, and
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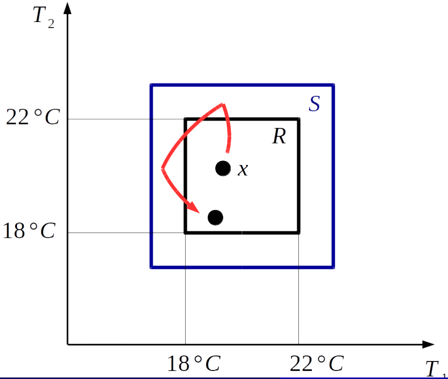

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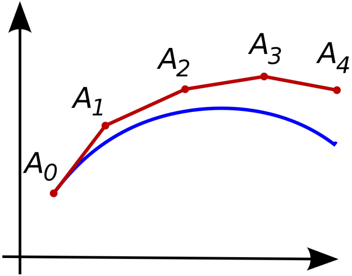
$\Rightarrow$ Need to know $x(t)$

Euler's estimation method of $x(t)$ (with $\dot{x}(t) = f(x(t))$)

$$\tilde{x}(t) = \tilde{x}(t_0) + f(\tilde{x}(t_0))(t - t_0)$$



Suppose that, for the current step size τ (or a sub-sampling size h), the derivative is constant and equal to the derivative at the starting point

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$$error(t) \leq \frac{hM}{2L}(e^{L(t-t_0)} - 1)$$

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We now consider a more appropriate constant λ that leads to sharper estimations of the Euler error.

Dahlquist's constant λ ("one-sided Lipschitz" constant)

- $\lambda \in \mathbb{R}$ is a constant s.t., for all $x, y \in S$:

$$\langle f(y) - f(x), y - x \rangle \leq \lambda \|y - x\|^2$$

where $\langle \cdot, \cdot \rangle$ denote the scalar product of two vectors of $\mathbb{R}^n$

[1] Define $V(x, x') = \|x - x'\|^2$; we have: $\frac{dV}{dt} \leq \lambda V$ (hence $V = V_0 e^{\lambda t}$). So V is an exponentially stable Lyapunov function when $\lambda < 0$.

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- λ can be computed using constraint optimization algorithms

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Local error function $\delta(\cdot)$ estimated using constant λ

Given an initial error δ_0 of $\tilde{x}(t)$ (i.e.: $\|\tilde{x}(0) - x(0)\| \leq \delta_0$),
the local E. error fn $\delta(\cdot)$ (s.t.: $\|x(t) - \tilde{x}(t)\| \leq \delta(t)$, for $t \in [0, \tau]$)
can be defined (for each mode u) by:

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- if $\lambda < 0$:

$$\delta(t) = \left(\delta_0^2 e^{\lambda t} + \frac{C^2}{\lambda^2} \left(t^2 + \frac{2t}{\lambda} + \frac{2}{\lambda^2} \left(1 - e^{\lambda t} \right) \right) \right)^{\frac{1}{2}}$$

- if $\lambda = 0$:

$$\delta(t) = \left(\delta_0^2 e^{t} + C^2(-t^2 - 2t + 2(e^{t} - 1)) \right)^{\frac{1}{2}}$$

- if $\lambda > 0$:

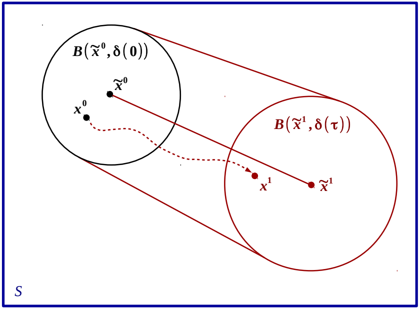
$$\delta(t) = \left(\delta_0^2 e^{3\lambda t} + \frac{C^2}{3\lambda^2} \left(-t^2 - \frac{2t}{3\lambda} + \frac{2}{9\lambda^2} \left(e^{3\lambda t} - 1 \right) \right) \right)^{\frac{1}{2}}$$

with $C = \sup_{x \in S} \ L\|f(x)\|$.

One-step invariance using the E. error fn $\delta(\cdot)$

- Given a ball $B^0 \equiv B(\tilde{x}^0, \delta^0) \subset S$, find a mode u s.t.:

$$x(t) \in S \quad \text{for all } x(0) \in B^0, t \in [0, \tau]$$

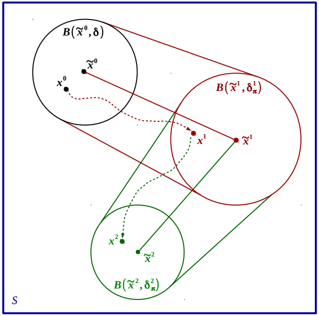


i.e.: $B^1 \equiv B(\tilde{x}^1, \delta^1) \subset S$ with $\tilde{x}^1 = \tilde{x}^0 + f(\tilde{x}^0)\tau$ and $\delta^1 = \delta(\tau)$ (assuming convexity of $\delta(\cdot)$ on $[0, \tau]$).

Finding a control pattern π using $\delta(\cdot)$

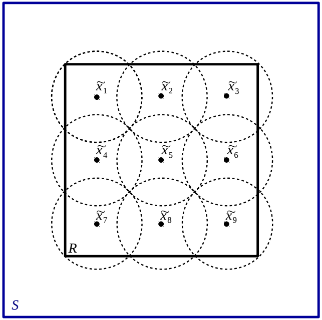
- Given a ball $B^0 \equiv B(\tilde{x}^0, \delta^0) \subset S$, find a pattern π (of length k) s.t.:

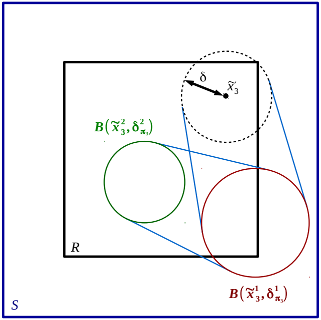
$$x(t) \in S \quad \text{for all } x(0) \in B^0, t \in [0, k\tau]$$



i.e.: $B^1 \equiv B(\tilde{x}^1, \delta^1) \subset S, \quad \ldots, \quad B^k \equiv B(\tilde{x}^k, \delta^k) \subset S$

(R,S)-stable control synthesis using E. error fn $\delta(\cdot)$





For each ball $B_i^0 \equiv B(\tilde{x}_i^0, \delta_i^0) \subset S$ covering R, find a pattern π_i (of length k_i) s.t.:

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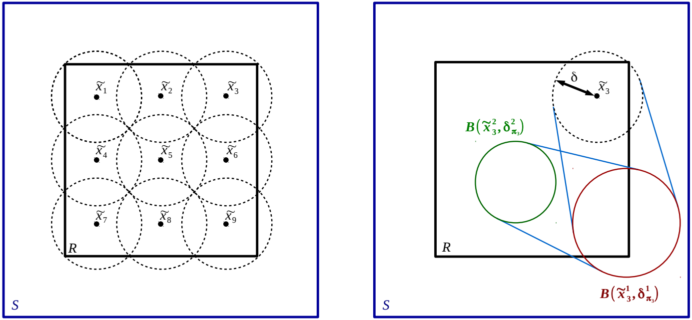

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- Safety: $B_i^1 \equiv B(\tilde{x}_i^1, \delta_i^1) \subset S, ..., B_i^{k_i-1} \equiv B(\tilde{x}_i^{k_i-1}, \delta_i^{k_i-1}) \subset S$, and

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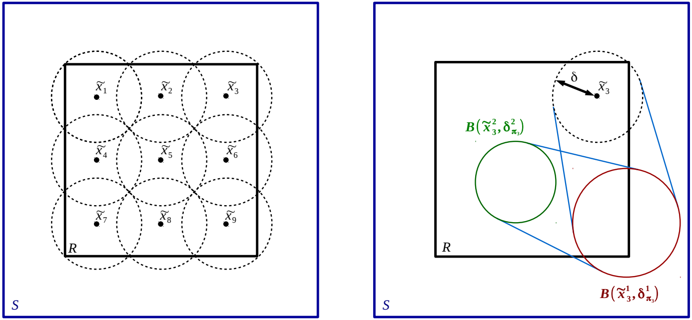

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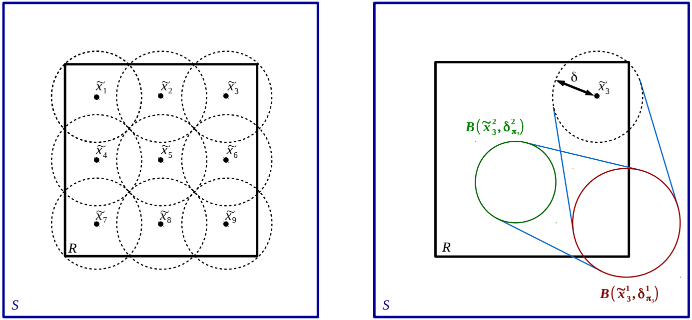

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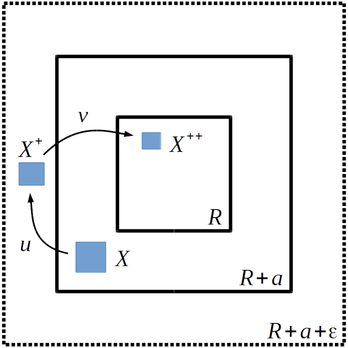
$\Rightarrow$ Combinatorial problem...

Illustration of Distributed vs. Centralized Control

Centralized control synthesis

$$\dot{x}(t) = f_u(x(t))$$

Example of a validated pattern of length 2 mapping the "ball" X into R with $S = R + a + \varepsilon$ as safety box:



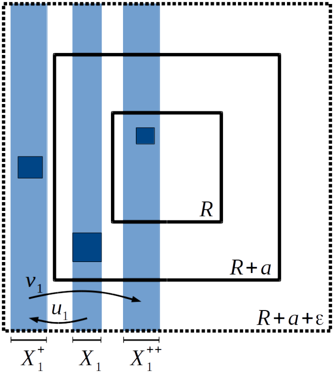
- $X \subset R$
- $X^+ = f_u(X) \subset S$
- $X^{++} = f_v(X^+) \subset R$

- Pattern $u \cdot v$ depends on X

Distrib. Control Synth. (of x_1 using S_2 as approx. of x_2)

$$\dot{x}_1(t) = f^1_{u_1}(x_1(t), x_2(t))$$
$$\dot{x}_2(t) = f^2_{u_2}(x_1(t), x_2(t))$$

Target zone: $R = R_1 \times R_2$



- $X_1 \subset R_1$
- $X_1^+ = f^1_{u_1}(X_1, S_2) \subset S_1$
- $X_1^{++} = f^1_{v_1}(X_1^+, S_2) \subset R_1$

- Pattern $u_1 \cdot v_1$ depends only on X_1

Distrib. Control Synth. (of x_2 using S_1 as approx. of x_1)

$$\dot{x}_1(t) = f^1_{u_1}(x_1(t), x_2(t))$$
$$\dot{x}_2(t) = f^2_{u_2}(x_1(t), x_2(t))$$

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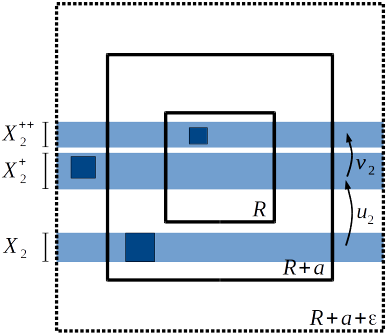
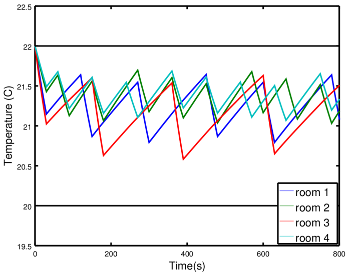
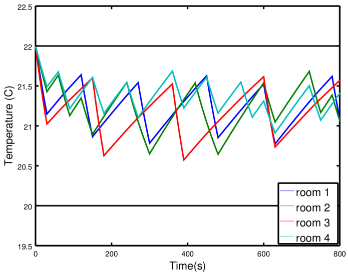


- $X_2 \subset R_2$
- $X_2^+ = f^2_{u_2}(S_1, X_2) \in S_2$
- $X_2^{++} = f^2_{v_2}(S_1, X_2^+) \in R_2$
- Pattern $u_2 \cdot v_2$ depends only on X_2

Application to distributed control of switched systems





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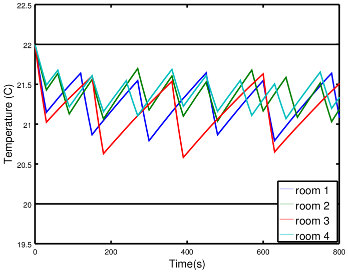
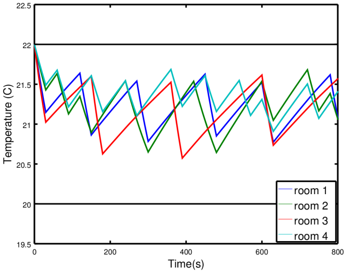
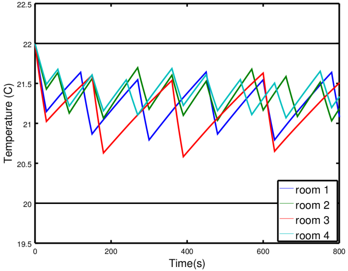
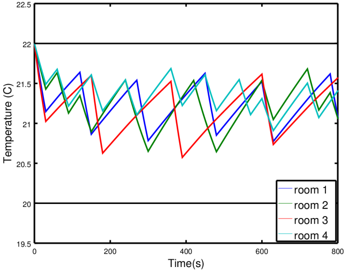




- centralized synthesis ($|\pi| = 2$): sub-sampling $h = \frac{\tau}{20}$,
  2^4 modes, 256 balls $\rightarrow$ 48 s. of CPU time.

Application to distributed control of switched systems





- centralized synthesis ($|\pi| = 2$): sub-sampling $h = \frac{\tau}{20}$, 2^4 modes, 256 balls $\rightarrow$ 48 s. of CPU time.

- distributed synthesis ($|\pi| = 2$): sub-sampling $h = \frac{\tau}{10} \mid h = \frac{\tau}{1}$, $2^2 \mid 2^2$ modes, 16 | 16 balls $\rightarrow$ < 1 s. of CPU time.

Final remarks

1. Very simple method
2. Easy to implement (a few hundreds of lines of Octave)
3. Fast, but may lack precision w.r.t. sophisticated refinements of interval-based methods (even in the context of control synthesis)
4. Method can be adapted to guarantee reachability (instead of stability)
5. Replacement of forward Euler's method by better numerical schemes (e.g.: backward Euler, Runge-Kutta of order 4) does not seem to yield significant gain
6. Several examples for which E.-based control synthesis
   - beats state-of-art interval-based control methods (e.g.: 4-room building ventilation)
   - fails/ is beaten by standard interval-based control methods (e.g.: DC-DC Boost converter)