Membership problem in $\mathrm{GL}(2,\mathbb{Z})$ extended by singular matrices

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joint work with Igor Potapov

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Membership problem

Let M be an $n \times n$ matrix and $\mathcal{F} = \{M_1, \ldots, M_k\}$ be a finite collection of $n \times n$ matrices. Determine whether $M \in \langle \mathcal{F} \rangle$, that is, whether

$$M = M_{i_1} M_{i_2} \cdots M_{i_t}$$

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In case when M is the zero matrix, the above problem is called the mortality problem.

Known results

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- Membership problem is decidable in PTIME for commuting matrices (over algebraic numbers) [Babai, et. al., 1996]
- It is a long standing open question whether the membership problem is decidable for 2×2 matrices (even over integers).

Known results

Let $\mathrm{GL}(2,\mathbb{Z}) = \{A \in \mathbb{Z}^{2\times 2} \ : \ \det(A) = \pm 1\}$.
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- The membership problem is decidable for 2×2 nonsingular integer matrices. [P. Semukhin and I. Potapov, 2017]
- The mortality problem is decidable for 2×2 integer matrices with determinant $0, \pm 1$ (i.e. for matrices from $\mathrm{GL}(2,\mathbb{Z})$ and singular matrices) [C. Nuccio and E. Rodaro, 2008]

Main result

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The membership problem is decidable for 2×2 integer matrices with determinant $0, \pm 1$ (i.e. for matrices from $\mathrm{GL}(2, \mathbb{Z})$ and singular matrices)

Theorem (Smith Normal Form)

For any matrix $A \in \mathbb{Z}^{2\times 2}$, *there are matrices* E, F *from* $\mathrm{GL}(2,\mathbb{Z})$ *such that* $A = E \begin{bmatrix} m & 0 \\ 0 & nm \end{bmatrix} F$ *for some* $n, m \in \mathbb{N} \cup \{0\}$.

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If $A \in \mathbb{Z}^{2\times 2}$ is a singular matrix, then the Smith normal form of A is equal to $\begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix}$, where t is the gcd of the coefficients of A.

Theorem

Given a singular 2×2 integer matrix M and a set $\mathcal{F} = \{A_1, \ldots, A_n, B_1, \ldots, B_m\}$, where $A_1, \ldots, A_n \in \mathrm{GL}(2, \mathbb{Z})$ and $B_1, \ldots, B_m$ are 2×2 singular integer matrices.
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We will construct a graph $\mathcal{G}(M, \mathcal{F})$ with the property: $M \in \langle \mathcal{F} \rangle$ if and only if there is a path in $\mathcal{G}(M, \mathcal{F})$ from an initial to a final node of weight t.

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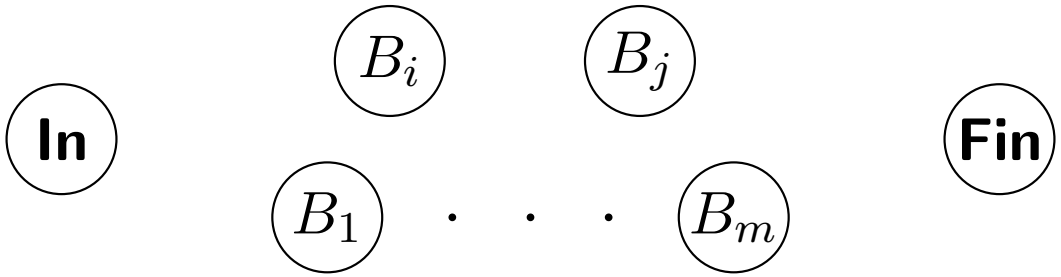
Graph $\mathcal{G}(M, \mathcal{F})$ has m nodes labelled by singular matrices $B_1, \ldots, B_m$ and two special nodes **In** and **Fin**.

If $B_i = E_i \begin{bmatrix} t_i & 0 \\ 0 & 0 \end{bmatrix} F_i$, then the weight of B_i is equal to t_i.

In and **Fin** have weight 1.

Description of $\mathcal{G}(M, \mathcal{F})$



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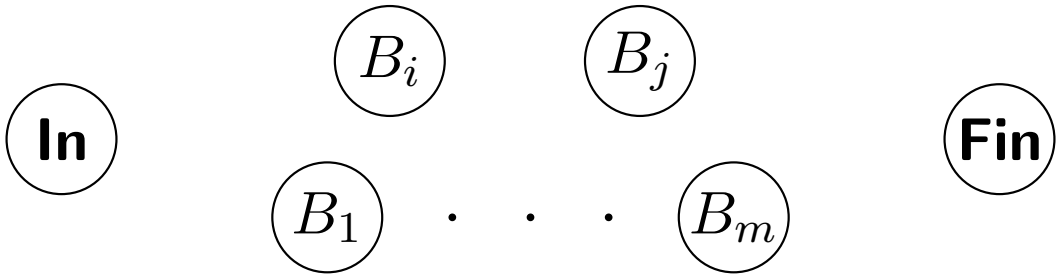

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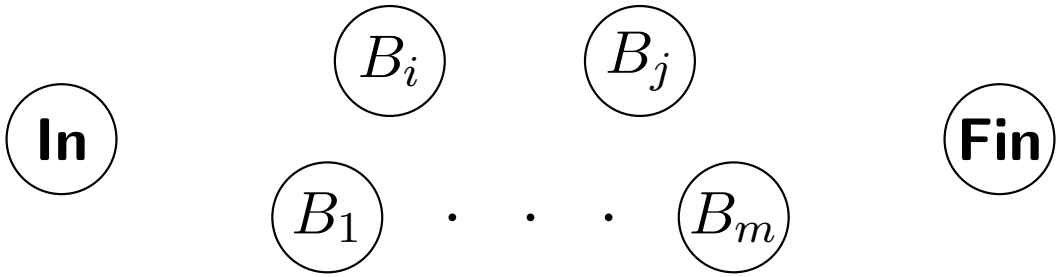

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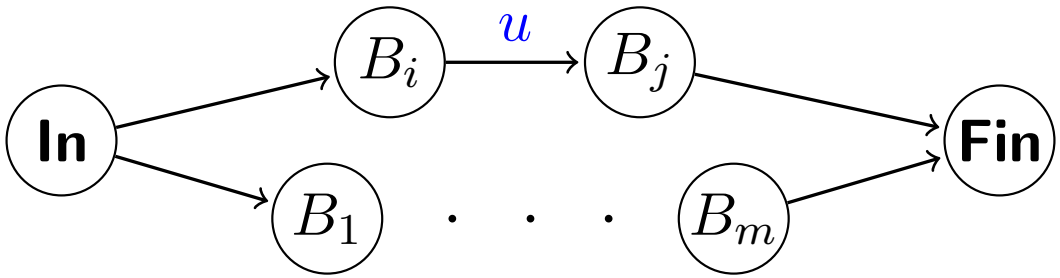

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For every u such that $-t \leq u \leq t$ we add an edge from B_i to B_j of weight u if and only if there is a matrix $C \in \langle A_1, \ldots, A_n \rangle$ such that $F_i C E_j = \begin{bmatrix} u & x \\ y & z \end{bmatrix}$, where $x, y, z \in \mathbb{Z}$.

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An edge $B_i \xrightarrow{u} B_j$ corresponds to a product

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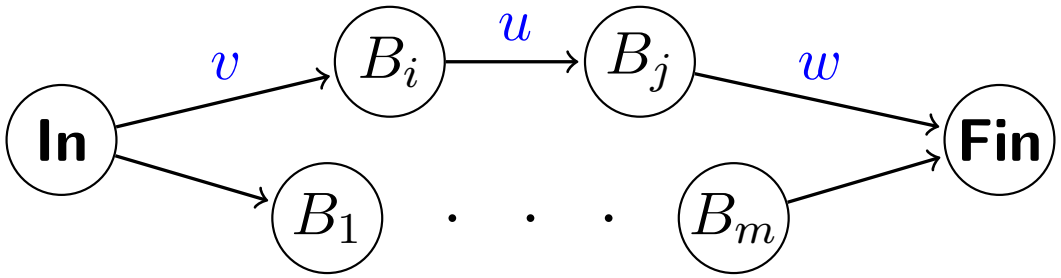

Recall that $M = E \begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix} F$ and $B_i = E_i \begin{bmatrix} t_i & 0 \\ 0 & 0 \end{bmatrix} F_i$.

We add an edge of weight v from **In** to B_i if there is a matrix $C \in \langle A_1, \ldots, A_n \rangle$ such that $E^{-1} C E_i = \begin{bmatrix} v & x \\ 0 & y \end{bmatrix}$, where $x, y \in \mathbb{Z}$.

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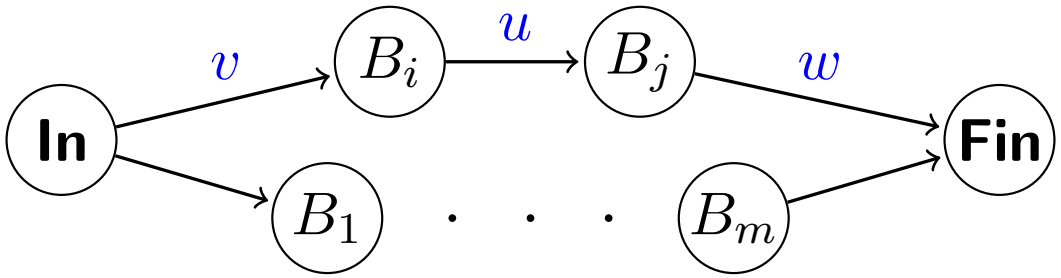

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We add an edge of weight w from B_j to **Fin** if there is a matrix $C \in \langle A_1, \ldots, A_n \rangle$ such that $F_j C F^{-1} = \begin{bmatrix} w & 0 \\ x & y \end{bmatrix}$, where $x, y \in \mathbb{Z}$.

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Proposition

Let $M = E \begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix} F$ be the Smith normal form of matrix M.
Then $M \in \langle \mathcal{F} \rangle$ if and only if there is a path in $\mathcal{G}(M, \mathcal{F})$ from **In** to **Fin** of weight t.

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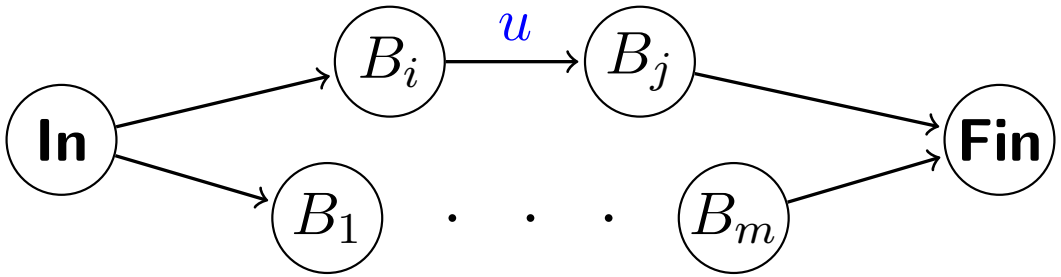
Proposition

If there is a path in $\mathcal{G}(M, \mathcal{F})$ from **In** to **Fin** of weight t, then there is such path of length at most $2m \log_2 t + 2m + \log_2 t$.



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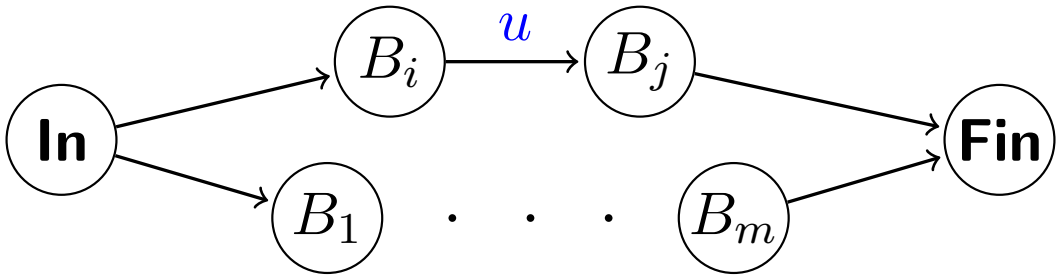

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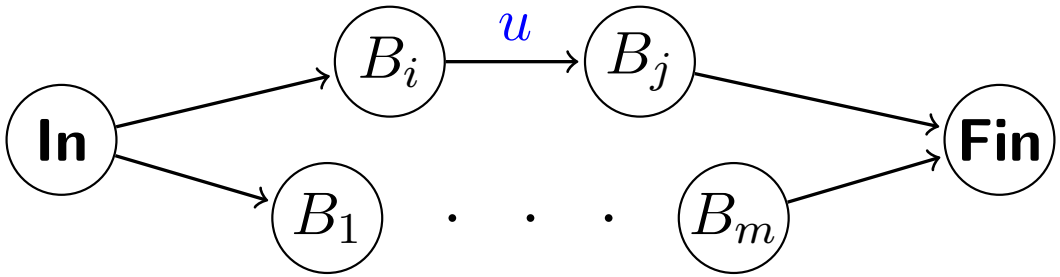
The group $\mathrm{GL}(2, \mathbb{Z})$ is generated by the matrices $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $R = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ and $N = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.



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So any matrix $A \in \mathrm{GL}(2, \mathbb{Z})$ is represented by a word in the alphabet $\Sigma = \{S, R, N\}$.

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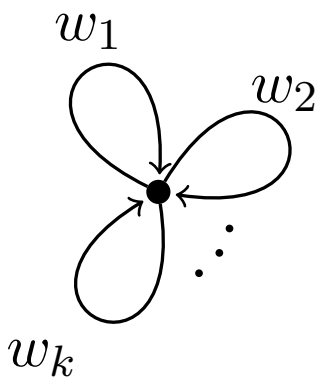

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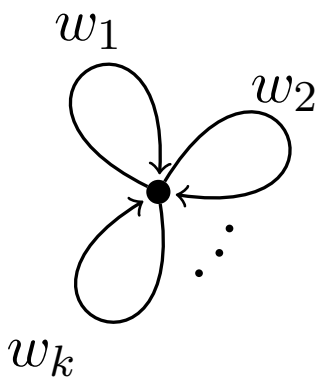

Theorem (Choffrut and Karhumäki, 2005)

Given two regular subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathrm{GL}(2,\mathbb{Z})$, it is decidable whether the intersection $\mathcal{S}_1 \cap \mathcal{S}_2$ is empty or not.

For every u such that $-t \leq u \leq t$ we add an edge from B_i to B_j of weight u if and only if there is a matrix $C \in \langle A_1, \ldots, A_n \rangle$ such that $F_i C E_j = \begin{bmatrix} u & x \\ y & z \end{bmatrix}$, where $x, y, z \in \mathbb{Z}$.

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Hence we can decide if there is an edge from B_i to B_j of weight u.

For every u such that $-t \leq u \leq t$ we add an edge from B_i to B_j of weight u if and only if there is a matrix $C \in \langle A_1, \ldots, A_n \rangle$ such that $F_i C E_j = \begin{bmatrix} u & x \\ y & z \end{bmatrix}$, where $x, y, z \in \mathbb{Z}$.

The set $\{F_i C E_j \ : \ C \in \langle A_1, \ldots, A_n \rangle\}$ is a regular subset of $\mathrm{GL}(2, \mathbb{Z})$.

For any fixed $u \in \mathbb{Z}$, the set $\left\{ \begin{bmatrix} u & x \\ y & z \end{bmatrix} \in \mathrm{GL}(2, \mathbb{Z}) \ : \ x, y, z \in \mathbb{Z} \right\}$ is a regular subset of $\mathrm{GL}(2, \mathbb{Z})$.

Hence we can decide if there is an edge from B_i to B_j of weight u.

THANK YOU.